

The Scattering amplitude for one parameter family of shape invariant potentials related to X_m Jacobi polynomials

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Abstract

We consider the recently discovered, one parameter family of exactly solvable shape invariant potentials which are isospectral to the generalized Pöschl-Teller potential. By explicitly considering the asymptotic behaviour of the X_m Jacobi polynomials associated with this system ($m = 1, 2, 3, \dots$), the scattering amplitude for the one parameter family of potentials is calculated explicitly.

1 Introduction

The ideas of Supersymmetric quantum mechanics (SQM) and shape invariant potential (SIP) have not only enriched our understanding of the exactly solvable potentials but have helped in substantially increasing the list of exactly solvable potentials [1]. In particular, the search for the exactly solvable potentials has been boosted greatly due to the recent discovery of exceptional orthogonal polynomials (EOP) (also known as X_m Laguerre and X_m Jacobi polynomials) [4, 2, 3]. Unlike the usual orthogonal polynomials, these EOPs start with degree $m \geq 1$ and still form a complete orthonormal set with respect to a positive definite innerproduct defined over a compact interval. This remarkable work lead Quesne [5] to the discovery of two new SIPs (with translation) whose solution is in terms of X_1 Laguerre and X_1 Jacobi polynomials. Soon afterwards, a third SIP (with translation) was discovered whose solution is also in terms of X_1 Jacobi polynomials [6]. Subsequently,

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Odake and Sasaki constructed three one parameter family of shape invariant potentials (with translation) whose bound state eigenfunctions are in terms of X_m Laguerre and X_m Jacobi polynomials [9]. It is worth reminding here that all of these are isospectral to the well known SIPs.

It is not usually appreciated that unlike the usual SIPs, the newly discovered SIPs are explicitly \hbar dependent. Further, while two out of the three newly discovered SIPs have pure bound state spectrum, the third SIP which is isospectral to the generalized Pöschl-Teller (GPT) potential, has both discrete and continuum spectrum. Recently, we have calculated the scattering amplitude for the SIP which is isospectral to GPT potential and whose bound state eigenfunction is in terms of X_1 Jacobi polynomial [8]. The purpose of the present paper is to extend this work to a class of isospectral potentials. In particular, in this paper we consider one parameter family of SIPs which are isospectral to GPT and whose bound state eigenfunction is given in terms of X_m Jacobi polynomials ($m = 1, 2, 3, \dots$) and obtain the scattering amplitude for this family by considering the asymptotic behaviour of the X_m exceptional Jacobi polynomials (EOP).

This paper is organized as follow: In Sec. 2, to set the notation, we briefly review the work of Odake and Sasaki [9] regarding the bound state eigenvalues and eigenfunctions for the one parameter family of SIPs which are isospectral to GPT and whose bound state eigenfunctions are in terms of X_m Jacobi polynomials. To motivate our calculation for the general case, the scattering amplitude for the potential with the bound state eigenfunction in terms of X_2 Jacobi polynomial is discussed in Sec. 3. The most general X_m case is discussed in Sec. 4. We summarize our conclusions in Sec. 5.

2 Bound State of Infinitely many shape invariant potentials

In this section we essentially set the notation by reviewing the work of Odake and Sasaki [9] regarding one parameter family SIPs and the corresponding bound states. We mostly adopt their notations in this paper.

For $m \geq 1$, the shape invariant prepotential $\omega_l(r; \lambda)$ which is isospectral to the GPT is given by

$$\omega_m(r; \lambda) = \omega_0(r; \lambda + m\delta) + \log \frac{\xi_m(\cosh 2r; \lambda + \delta)}{\xi_m(\cosh 2r; \lambda)}; \quad 0 \leq r \leq \infty \quad (1)$$

where $\lambda = (g, h)$, $h > g > 0$, $\delta = (1, -1)$, while $\xi_m(\cosh 2r; \lambda)$ is related with the Jacobi polynomial as follows:

$$\xi_m(\cosh 2r; \lambda) = P_m^{(-g-m-\frac{1}{2}, -h+m-\frac{3}{2})}(\cosh 2r). \quad (2)$$

It may be noted that the prepotential related with GPT corresponds to $m = 0$ and is given by

$$\omega_0(r; \lambda) = g \log \sinh r - h \log \cosh r. \quad (3)$$

The general Hamiltonian corresponding to the prepotential $\omega_m(r; \lambda)$ is given by

$$\mathcal{H}_m(\lambda) = \mathcal{A}_m(\lambda)^\dagger \mathcal{A}_m(\lambda) = p^2 + V_m(r), \quad (4)$$

where

$$p = -i\partial_r \quad V_m(r) = \omega'_m(r; \lambda)^2 + \omega''_m(r; \lambda), \quad (5)$$

$$\mathcal{A}_m(\lambda) = \partial_r - \omega'_m(r; \lambda), \quad \mathcal{A}_m(\lambda)^\dagger = -\partial_r - \omega'_m(r; \lambda). \quad (6)$$

Here prime on $\omega_l(r; \lambda)$ denotes derivative with respect to r . The prepotential $\omega_m(r; \lambda)$ satisfies the shape invariance condition

$$\mathcal{A}_m(\lambda) \mathcal{A}_m(\lambda)^\dagger = \mathcal{A}_m(\lambda + \delta) \mathcal{A}_m(\lambda + \delta)^\dagger + \mathcal{E}_1(\lambda + m\delta) \quad (7)$$

This further implies

$$\omega'_m(r; \lambda)^2 - \omega''_m(r; \lambda) = \omega'_m(r; \lambda + \delta)^2 + \omega''_m(r; \lambda + \delta) + \mathcal{E}_1(\lambda + m\delta), \quad (8)$$

in which δ is a certain shift of the parameter λ . The general form of entire set of discrete eigenvalues and corresponding eigenfunctions of $\mathcal{H}_m(\lambda)$ are obtained by solving,

$$\mathcal{H}_m(\lambda) \psi_{m,\nu}(r; \lambda) = \mathcal{E}_{m,\nu}(\lambda) \psi_{m,\nu}(r; \lambda) \quad (9)$$

The discrete eigenvalues are,

$$\mathcal{E}_{m,\nu}(\lambda) = \mathcal{E}_\nu(\lambda + m\delta) = \sum_{k=0}^{\nu-1} \mathcal{E}_1(\lambda + k\delta + m\delta) = 4\nu(h - g - 2m - \nu) \quad (10)$$

with $\nu = 0, 1, 2, \dots, \nu_B - m$; $\nu_B = \frac{(h-g)}{2}$. The corresponding eigenfunctions are written as

$$\begin{aligned} \psi_{m,\nu}(r; \lambda) &= \phi_m(r; \lambda) P_{m,\nu}(\cosh 2r; \lambda) \\ &= \frac{e^{\omega_0(r; \lambda + m\delta)}}{\xi_m(\cosh 2r; \lambda)} P_{m,\nu}(\cosh 2r; \lambda), \end{aligned} \quad (11)$$

with

$$P_{m,\nu}(\cosh 2r; \lambda) = a_{m,\nu}(r; \lambda) P_\nu(\cosh 2r; \lambda + m\delta) + b_{m,\nu}(r; \lambda) P_{\nu-1}(\cosh 2r; \lambda + m\delta). \quad (12)$$

Here the coefficients $a_{m,\nu}(r; \lambda)$ and $b_{m,\nu}(r; \lambda)$ are given by [9]

$$\begin{aligned} a_{m,\nu}(r; \lambda) = \xi_m(\cosh 2r; g+1, h-1) &+ \frac{2\nu(-g-h+m-1)\xi_{m-1}(r; g, h-2)}{(-g-h+2m-2)(g-h+2\nu+2m-1)} \\ &- \frac{\nu(-2h+4m-3)\xi_{m-2}(r; g+1, h-3)}{(2g+2\nu+1)(-g-h+2m-2)}, \end{aligned} \quad (13)$$

$$b_{m,\nu}(r; \lambda) = \frac{(-g - h + m - 1)(2g + 2\nu + 2m - 1)\xi_{m-1}(r; g, h - 2)}{(2g + 2\nu + 1)(g - h + 2\nu + 2m - 1)}. \quad (14)$$

It is worth noting that the polynomials $P_{m,\nu}(r; \lambda)$ are orthogonal with respect to the measure $\phi_m(r; \lambda)^2$, i.e.

$$\begin{aligned} & \int_0^\infty dr \phi_m(r; \lambda)^2 P_{m,\nu}(r; \lambda) P_{m,q}(r; \lambda) \\ &= h_{m,\nu}(g, h) \delta_{\nu m} = h_\nu(g + m, g - m) \frac{(\nu + g + m + \frac{1}{2})(h - \nu - 2m + \frac{1}{2})}{(\nu + g + \frac{1}{2})(h - \nu - m + \frac{1}{2})} \delta_{\nu q}, \end{aligned} \quad (15)$$

where

$$h_\nu(\lambda) = \frac{\Gamma(\nu + g + \frac{1}{2})\Gamma(h - g - \nu + 1)}{2\nu!(h - g - 2\nu)\Gamma(h - \nu + \frac{1}{2})}. \quad (16)$$

It is remarkable that even though the potential related with GPT, i.e. $V_{GPT} = \omega'_0(r; \lambda)^2 - \omega''_0(r; \lambda)$ is very different from the potential $V_m = \omega'_m(r; \lambda)^2 - \omega''_m(r; \lambda)$, the bound state spectrum(10) of the two for any integral m is still the same, however the corresponding eigenfunctions are different. Replacing $2r$ by r , and after using (2),(3) in (11), the bound state wave function related to the X_m Jacobi polynomial, is given by

$$\psi_\nu^m(r) = N_\nu^m \frac{(\cosh r - 1)^{\frac{1}{2}(\alpha+1/2)}(\cosh r + 1)^{\frac{1}{2}(\beta+1/2)}}{P_m^{(-\alpha-1, -\beta-1)}(\cosh r)} \hat{P}_{\nu+m}^{(\alpha, \beta)}(\cosh r) \quad (17)$$

where $\alpha = g + m - \frac{1}{2}$, $\beta = -h + m - \frac{1}{2}$, $N_\nu^m = [2^{(h-g-2m+1)} h_{m,n}(g, h)]^{\frac{1}{2}}$, is the normalization constant, $\hat{P}_{\nu+m}^{(\alpha, \beta, m)}(\cosh r)$ is $(\nu + m)$ th-degree X_m Jacobi Polynomial and $P_m^{(-\alpha-1, -\beta-1)}(\cosh r)$ is usual Jacobi polynomial.

3 Calculation of scattering amplitude for $m=2$ (X_2 Jacobi polynomial)

The relation between the X_m Jacobi polynomial and the usual Jacobi polynomial is given by [7]

$$\begin{aligned} P_{m,\nu}(\cosh r) &= \hat{P}_{\nu+m}^{(\alpha, \beta)}(\cosh r) = \left(P_m^{(-\alpha-2, \beta)}(\cosh r) + \frac{2\nu(m - \alpha + \beta - 1)P_{m-1}^{-\alpha, \beta}(\cosh r)}{(2m - \alpha + \beta - 2)(2\nu + \alpha + \beta)} \right. \\ &\quad \left. - \frac{\nu(\beta + m - 1)P_{m-2}^{(-\alpha, \beta)}(\cosh r)}{(\alpha + \nu - m + 1)(2m - \alpha + \beta - 2)} P_\nu^{(\alpha, \beta)}(\cosh r) \right) \\ &\quad + \frac{(m - \alpha + \beta - 1)(\alpha + \nu)}{(\alpha + \nu - m + 1)(2\nu + \alpha + \beta)} P_{m-1}^{(-\alpha, \beta)}(\cosh r) P_{n-1}^{(\alpha, \beta)}(\cosh r) \end{aligned} \quad (18)$$

where

$$P_{\nu}^{(\alpha, \beta)}(\cosh r) = \frac{\Gamma(\alpha + \nu + 1)}{\nu! \Gamma(\alpha + \beta + l + 1)} \sum_{q=1}^{\nu} \binom{\nu}{p} \frac{\Gamma(\alpha + \beta + \nu + p + 1)}{\Gamma(\alpha + p + 1)} \left(\frac{\cosh r - 1}{2} \right)^q \quad (19)$$

Using this relation, we have recently calculated the scattering amplitude for the $m = 1$ (X_1 Jacobi case) [8]. We now extend that discussion to the $m = 2$ case. For X_2 Jacobi case, we set $m = 2$ in the above expression, to get

$$\begin{aligned} \hat{P}_{\nu+2}^{(\alpha, \beta)}(\cosh r) &= \left[\frac{1}{2} \{ \alpha(\beta + 2) + (\alpha - \beta - 2)(\alpha - \beta - 1) \} - \frac{(\alpha - \beta - 1)(\beta - \alpha + 2)}{8} x^2 \right. \\ &+ \left(\frac{(\alpha - \beta - 1)(\alpha + \beta + 2)}{4} - \frac{\nu(\beta - \alpha + 1)(\alpha - \beta - 2)}{(\beta - \alpha + 2)(\beta + \alpha + 2\nu)} \right) x \\ &- \frac{\nu(\beta - \alpha + 1)(\alpha + \beta)}{(\beta - \alpha + 2)(\beta + \alpha + 2\nu)} - \frac{\nu(\beta + 1)}{(\alpha + \nu - 1)(\beta - \alpha + 2)} \Big] P_{\nu}^{(\alpha, \beta)}(\cosh r) \\ &- \frac{(\beta - \alpha + 1)(\alpha + \nu)}{2(\beta + \nu - 1)(\alpha + \beta + 2\nu)} [(\alpha + \beta - 2)x + (\alpha + \beta)] P_{\nu-1}^{(\alpha, \beta)}(\cosh r) \quad (20) \end{aligned}$$

The usual Jacobi polynomial $P_{\nu}^{(\alpha, \beta)}(\cosh r)$ can be written in terms of Hypergeometric function as :

$$P_{\nu}^{(\alpha, \beta)}(\cosh r) = \frac{\Gamma(\nu + \alpha + 1)}{\nu! \Gamma(1 + \alpha)} F(\nu + \alpha + \beta + 1, -\nu, 1 + \alpha; \frac{1 - \cosh r}{2}). \quad (21)$$

To get the scattering state wave functions for this system, two modifications of the bound state wavefunctions are required [10]: (i) The second solution of the Schrödinger equation which diverges asymptotically and hence had been discarded earlier, must be retained. (ii) The discrete level ν should be replaced by the wavenumber k such that one gets asymptotic behavior in terms of $e^{\pm ikr}$ as $r \rightarrow \infty$.

Equation(21) can be written by considering the second solution as,

$$\begin{aligned} P_{\nu}^{(\alpha, \beta)}(\cosh r) &= \frac{\Gamma(\nu + \alpha + 1)}{\nu! \Gamma(1 + \alpha)} \left[C_1 F(\nu + \alpha + \beta + 1, -\nu, 1 + \alpha; \frac{1 - \cosh r}{2}) \right. \\ &+ \left. C_2 \left(\frac{1 - \cosh r}{2} \right)^{-(\nu + \alpha + \beta + 1)} F(\nu + \beta + 1, -\nu - \alpha, 1 - \alpha; \frac{1 - \cosh r}{2}) \right] \quad (22) \end{aligned}$$

We consider the boundary condition, $r \rightarrow 0$, i.e. $(\frac{1 - \cosh r}{2}) \rightarrow 0$, $\psi_{\nu}(r) \rightarrow \text{finite}$, the allowed solution is

$$P_{\nu}^{(\alpha, \beta)}(\cosh r) = \frac{\Gamma(\nu + \alpha + 1)}{\nu! \Gamma(1 + \alpha)} C_1 F(\nu + \alpha + \beta + 1, -\nu, 1 + \alpha; \frac{1 - \cosh r}{2}) \quad (23)$$

where C_1 is a constant. In order to compare our results with the previous results [8], we use $\alpha = B - A - \frac{1}{2}$, $\beta = -B - A - \frac{1}{2}$. Now replacing ν by $A + ik$, we get

$$P_{(A+ik)}^{(\alpha, \beta)}(\cosh r) = C_1 \frac{\Gamma(B + ik + 1/2)}{(A + ik)! \Gamma(B - A + 1/2)} F(-A + ik, -A - ik, B - A + 1/2; \frac{1 - \cosh r}{2}), \quad (24)$$

$$P_{(A+ik-1)}^{(\alpha,\beta)}(\cosh r) = C_1 \frac{\Gamma(B+ik-1/2)}{(A+ik-1)!\Gamma(B-A+1/2)} \times F(-A+ik-1, -A-ik+1, B-A+1/2; \frac{1-\cosh r}{2}). \quad (25)$$

Using Eqs. (24) and (25) in (20) we get $\hat{P}_{(\nu+2)}^{(\alpha,\beta)}(\cosh r) = \hat{P}_{(A+ik+2)}^{(\alpha,\beta)}(\cosh r)$.

The scattering state wavefunction thus is given by

$$\psi_k(r) =$$

$$N_k^2 \frac{(\cosh r - 1)^{\frac{1}{2}(B-A)} (\cosh r + 1)^{-\frac{1}{2}(B+A)} P_{A+ik+1}^{(\alpha,\beta)}(\cosh r)}{(2(B-1)(2B-1) \cosh^2 r + 2(2B+1)(2A+1) \cosh r + 4A^2 + 4A + 2B - 1)} \quad (26)$$

Using the properties of hypergeometric function [11],

$$F(\alpha, \beta, \gamma; z) = (1-z)^{-\alpha} \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} F(\alpha, \gamma-\beta, \alpha-\beta+1; \frac{1}{1-z}) + (1-z)^{-\beta} \frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)} F(\beta, \gamma-\alpha, \beta-\alpha+1; \frac{1}{1-z}), \quad (27)$$

and taking the limit $r \rightarrow \infty$, finally we get the asymptotic form of (26), as

$$\lim_{r \rightarrow \infty} \psi_k(r) = N_k^2 \frac{C_1 P 2^{A+1} 4^{ik}}{16} \left[\left(\frac{ac}{P} \right) 2^{-4ik} e^{ikr} + e^{-ikr} \right], \quad (28)$$

where

$$P = \frac{(B+ik-3/2)(2ik-1)(ab) + 2(B+ik-1/2)(ed)}{(B+ik-3/2)(2ik-1)}; \quad a = \frac{\Gamma(B+ik+1/2)}{(A+ik)!\Gamma(B-A+1/2)};$$

$$b = \frac{\Gamma(B-A+1/2)\Gamma(-2ik)}{\Gamma(-A-ik)\Gamma(B-ik+1/2)}; \quad c = \frac{\Gamma(B-A+1/2)\Gamma(2ik)}{\Gamma(-A+ik)\Gamma(B+ik+1/2)};$$

$$d = \frac{\Gamma(B+ik-1/2)}{(A+ik-1)!\Gamma(B-A+1/2)}; \quad e = \frac{\Gamma(B-A+1/2)\Gamma(-2ik+2)}{\Gamma(-A-ik+1)\Gamma(B-ik+3/2)};$$

The asymptotic behavior for the radial wavefunction (for $l=0$) is given by [1]

$$\lim_{r \rightarrow \infty} \psi_k(r) \simeq \frac{1}{2k} [S_{l=0} e^{ikr} - e^{-ikr}] \quad (29)$$

From (28) and (29) we get

$$S_{l=0} = \left(\frac{ac}{P} \right) 2^{-4ik} \quad (30)$$

Using the values of P, a, b, c, d and e, we obtain the scattering amplitude for the X_2 Jacobi case

$$\begin{aligned}
S_{l=0} &= S_{l=0}^{GPT} \left[\frac{\{B^2 - (ik - 1/2)^2\} - (B - ik + 1/2)}{\{B^2 - (ik + 1/2)^2\} - (B + ik + 1/2)} \right] \\
&= \frac{\Gamma(2ik)\Gamma(-A - ik)\Gamma(B - ik + 1/2)2^{-4ik}}{\Gamma(-A + ik)\Gamma(-2ik)\Gamma(B + ik + 1/2)} \times \\
&\quad \left[\frac{\{B^2 - (ik - 1/2)^2\} - (B - ik + 1/2)}{\{B^2 - (ik + 1/2)^2\} - (B + ik + 1/2)} \right]
\end{aligned} \tag{31}$$

4 Calculation of scattering amplitude for X_m Jacobi case

We now proceed to generalize this calculation to the X_m case ($m = 1, 2, 3, \dots$). Following the calculation done above for the X_2 case, using Eqs. (19) and (23) in Eq. (18) and replacing $\nu \rightarrow A + ik$ we get $\hat{P}_{(\nu+m)}^{(\alpha,\beta)}(\cosh r) = \hat{P}_{(A+ik+m)}^{(\alpha,\beta)}(\cosh r)$. Now using $\hat{P}_{(A+ik+m)}^{(\alpha,\beta)}(\cosh r)$ and then taking the limit $r \rightarrow 0$, we obtain the asymptotic form of the wave function (17) for the X_m case

$$\lim_{r \rightarrow \infty} \psi_k(r) = N_k^m \frac{C_1 \Gamma(-2B + m - 1) \Gamma(-2B + 2m - 1) 4^{-A-2m+ik} P}{\Gamma(-3B + A + 2m - 1/2) \Gamma(-2B + m - 1)} \left[\left(\frac{ac}{Q} \right) 2^{-4ik} e^{ikr} + e^{-ikr} \right], \tag{32}$$

where

$$Q = (ab) + \frac{m(m - 2B - 1)(B + ik - 1/2)}{(B + ik - im + 1/2)(2ik - 1)}(ed), \tag{33}$$

while a, b, c, d, and e, are same as in the X_2 case. From (32) and (29) we get

$$S_{l=0} = \left(\frac{ac}{Q} \right) 2^{-4ik} \tag{34}$$

Using Q, a, b, c, d and e as given above, we finally have the expression for the scattering amplitude in the X_m case

$$\begin{aligned}
S_{l=0} &= S_{l=0}^{GPT} \left[\frac{\{B^2 - (ik - 1/2)^2\} + (B - ik + 1/2)(1 - m)}{\{B^2 - (ik + 1/2)^2\} + (B + ik + 1/2)(1 - m)} \right] \\
&= \frac{\Gamma(2ik)\Gamma(-A - ik)\Gamma(B - ik + 1/2)2^{-4ik}}{\Gamma(-A + ik)\Gamma(-2ik)\Gamma(B + ik + 1/2)} \times \\
&\quad \left[\frac{\{B^2 - (ik - 1/2)^2\} + (B - ik + 1/2)(1 - m)}{\{B^2 - (ik + 1/2)^2\} + (B + ik + 1/2)(1 - m)} \right]
\end{aligned} \tag{35}$$

As expected, in the special case of $m = 1$ and 2 we get back the expressions for the scattering amplitude as obtained in [8] and in Sec. III above, thereby providing a powerful

check on the calculations. Remarkably, in the limit $m = 0$, the scattering amplitude as given by Eq. (35) reduces to $S_{l=0}^{GPT}$, providing a further check on the calculations. It is amusing to note that as one goes from $m = 1$ to arbitrary integer value, there is simply a change by a factor of $(1 - m)$ in the second term in both the numerator and the denominator.

5 Summary

In this paper we have calculated the scattering amplitude for one parameter family of potentials (isospectral to GPT), whose bound state eigenfunctions are given in terms of X_m Jacobi polynomials. The bound state eigenvalues and eigenfunctions for these potentials were known before [9]. Thus, with the calculation in this paper, one now has a complete knowledge about both the bound state spectrum and the scattering amplitude for the one parameter family of SIPs which are isospectral to GPT.

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